A Study of Effect of Pollution on the Dynamics of Predator-Prey Interactions

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1. Introduction

The dynamic relation between predators and their prey has long been and will continue to be one of the important themes in both ecology and mathematical ecology due to its widespread existence and importance.

In environment, prey- predator interactions at different trophic levels form food chains and food webs. These food chains are controlled by internal and external factors. External factors are climate, soil, time etc. Internal factors are set of organisms that are present in the environment. Besides internal factors the balance of the external factors is important for the survival of the species. Due to interactions and natural factors like death and growth there is continuous change in the interacting populations. So the populations are continuously moving away from one state to the other state.

The twentieth century had seen a tremendous growth in industry, agriculture and urbanization, providing a lot of comforts to humans life style, but rapidly increasing non degradable garbage, industrial fumes, use of pesticides and chemical fertilizers and toxic waste is polluting the air, water and landscapes as well, thus contaminating the ecosystems around the world - from pole to pole, from the highest mountains to the ocean deep. These pollutants coming in many forms—organic, inorganic, even radioactive etc, can make life difficult, if not impossible, for humans, animals and other organisms alike.

When toxic chemicals and metals enter the environment, organisms may absorb them through their skin or ingest them in their food or water. Animals higher in the **food chain** accumulate these toxins in higher and higher concentrations, a process called **biomagnification**. Top predators-including fish, birds, and mammals-can have much higher levels of these toxins in their bodies, making them more likely to experience the diseases, birth defects, genetic mutations, and other deleterious effects of these poisons.

In order to use and regulate toxic substance wisely, we must assess the risk of the populations exposed to toxicant. Mathematical population model is a tool which can quantitatively analyze the gravity of situa-

tion by studying its effect on interacting species present in any environment. Lotka (1925) and Volterra (1927) were the first to propose a simple differential equation model of prey- predator interaction called the Lotka–Volterra model. Since then, many mathematical models have been formed based on realistic explicit and implicit biological assumptions

The problem of estimating qualitatively the effect of a toxicant on a population by mathematical models began in the early 1980s. For a general class of single population models with toxicant stress, **Ma et al. [1](1989)** obtained a survival threshold distinguishing between persistence in the mean end extinction of a single population under the hypothesis that the capacity of the environment is large relative to the population biomass, and that the exogenous input of toxicant into the environment is bounded. In **1987,Ma and Hallam [2]** studied two dimensional non-autonomous Lotka–Volterra model by the average method and obtained sufficient conditions for persistence and for extinction of the population. The threshold of the survival for a system of two species in a polluted environment was studied by **Huaping and Ma [3] (1990).** Population toxicant coupling has been applied in several contexts including Lotka–Volterra and chemostat like environments, resulting in ordinary, integro-differential and stochastic models. All these studies rely on the hypothesis of a complete spatially homogeneous environment. Recently, a spatial structure has been carried out by **Zhan Li et al. [4] (2004)** when a diffusive–convective model is proposed to describe the dynamics of a population in a polluted environment and the sufficient criteria on persistence and extinction of the population are derived.

From the literature surveyed by us it was observed that the models in polluted environment developed so far contain at most two species whereas in nature there are examples of more than two interacting populations. For example **cheetahs** and **lions** both feed on similar prey; they are negatively impacted by the presence of the other because they will have less food, and however they still persist together.

Sometimes two populations at lowest trophic level may compete for their survival. For example, **plants** consume nitrogen by absorbing it into their roots, making nitrogen unavailable to nearby plants. Plants

that produce many roots typically reduce **soil nitrogen** to very low levels, eventually killing neighboring plants. Woodpeckers and squirrels often compete for nesting rights in the same holes and spaces in trees.

Motivation for Study

The earth, its ecosystems, and its creatures are all deeply connected. Thus, the existence of many species depends on the survival of others, and human beings are not exceptions. There are many animals which actually help people just by performing their natural roles in their environment; like honey bee pollinates the flowers; ants and earthworms aid in decomposition and turn more soil etc.

Due to uncontrolled industrialization and urbanization our natural environmental conditions are being greatly affected. Due to which ecological balances are getting disturbed. Which in turn are affecting the lives of many species in different ways, like many species lost their habitats and further they lost their existence. For example we observe that the most commonly found sparrow is vanishing from our surroundings. Thus it is the need of the time that ecologists and mathematicians should study the past records, influencing factors and monitors the present situation by developing some guidelines which can helpful in maintaining the balance of our ecosystem. So that our future generations should not be deprived off the natural resources and coexistence of the man and a vast variety of species on this beautiful earth.

As the unplanned human activities like over usage of chemical fertilizers have some disastrous results by creating hypoxia in wet lands i.e. “dead zone” where fish cannot live, it is the need of the time to form models which can help in indicating warning limit of pollution in a habitat, so that its inhabitants might be saved by taking necessary precautions.

2. Some Related Terms

The definitions of some related terms are as follows:

2.1. Prey [5]: A prey is an organism that is seized by an organism to be eaten.

2.2. Predator [5]: A predator is an organism that depends on predation for its food.

2.3. Predator Functional Response [6]: Predator functional response to prey density refers to the change in the density of prey attacked per unit of time per predator as the prey density changes.

2.4. Competition: Struggle between organisms to survive in a habitat with limited resources.

2.5. Mutualism [7]: These are situations in which the interaction of two species is mutually beneficial.

2.6. Refuge: It is a concept which revolves around the escape of an organism from predation.

2.7. Time delay [8]: Time delay is inherent in the predator-prey systems. Both the predator and prey require their own times to grow from birth to maturity and from maturity to death.

2.8. Harvesting: To cut and gather a crop;To catch a number of animals or fish to eat.

2.9. Persistence: The existence of species.

3. Basic Preliminaries

The basic theorems or preliminaries helpful in determining the solution and its behavior are as follows:

3.1. Cauchy-Peano Existence Theorem[9] -: Consider a system of differential equations

$$x' = f(t,x), \quad x(t_0) = x_0 \qquad \text{(c)}$$

Where x is n dimensional vector and t is a scalar

let R be the rectangle , R : $| t - t_0 | \leq a$, $\| x - x_0 \| \leq b$ $(a, b > 0)$ let f be a continuous over R and let M = max $\| f(t,x) \|$, $(t,x) \in R$ and let $\alpha = \min (a , {}^{b}/_{M})$ then there exists a differential function

$\phi(t)$ which is a solution of (c) on the interval $\mid t - t_0 \mid \leq \alpha$ and satisfies the initial condition ,

$$x(t_0) = x_0$$

3.2. Lipschitz Condition[10] -: A function g satisfying the following inequality

$$|g(t,u_1) - g(t,u_2)| \leq K|\, u_1 - u_2 \,|$$

For all (t,u_1) , (t,u_2) in the region D is said to satisfy a Lipschitz condition in D and K is the Lipschitz constant.

Consider a system $\dot{x} = g(t,x)$, $x(t_0) = x_0$

If g(t,x) is a function of t and x in a closed and bounded region R(a,b) given as:

$$R(a,b): |\, t - t_0| \leq a, |\, x - x_0| \leq b$$

and satisfies the lipschitz condition in R(a,b), then ∃ a unique solution x(t) to the initial value problem defined on the interval $J : |t - t_0| \leq h$

Where h = min (a ,b/M)

And $|g(t,x)| \leq M$

3.3. Equilibrium Points[10] -: Consider the system ;

$$\dot{x} = f(t,x), \quad x \in R^n$$

Then any point $x_0 \in R^n$ at which $f(t,x)$ vanishes $\forall\, t \geq t_0$, is called an equilibrium point.

3.4. Stability [10]: Consider a system $\dot{x} = f(x)$, $x \in R^n$ **(a)**

Let $x^*(t)$ be the equilibrium solution of **(a)** then $x^*(t)$ is said to be stable if solution starting sufficiently close to $x^*(t)$ at a given time remain close to $x^*(t)$ for all later times.

3.5. Asymptotically stable [10] : Consider a system $\dot{x} = f(x)$, $x \in R^n$ (**a**)

Let $x^*(t)$ be the solution of (**a**) then $x^*(t)$ is said to be asymptotically stable if it is stable and nearby solutions of $x^*(t)$ actually converges to $x^*(t)$ as $t \to \infty$

3.6. Lyapunov Stability [10] - Consider the following vector field

$$\dot{x} = F(x) , \quad x \in R^n \qquad (\mathbf{a})$$

Let $x^* \in R^n$ be a equilibrium point of (**a**) and let

V: U → R be a C^1 function defined on some neighborhood U of x^* such that

i. V(x^*) = 0 and V(x) > 0 if $x \neq x^*$

ii. $V'(x) \leq 0$ in U –x^*

Then x^* is stable

iii. $V'(x) < 0$ in U – $\{x^*\}$

Then x^* is asymptotically stable

If U= R^n and (i) & (iii) hold then x^* is said to be globally asymptotically stable.

4. Literature Review

Since the advent of historic **Lotka and Volterra(1925-26)** model, this field has attracted many mathematicians who not only improved their model but also studied and added many more aspects affecting the dynamics of populations.

There are various related studies which are given below.

Albrectz et.al. (1973)[11] discussed an outstanding problem in mathematical modeling of ecological system that is of determining condition which guarantees the **uniqueness of limit cycles** in Prey –Predator model.

Mynard Smith (1974) [12] showed that the **presence of a constant proportion refuge** does not alter the dynamic stability of the neutrally stable Lotka – Volterra model , while **a constant number of refuge** of any size replaces the neutrally stable behavior with the stable equilibrium.

Cushing et.al. (1976) [13] took a general (lotka-volterra type) integro-differential system which described the predator-prey interaction. The **effect of delay** was considered. Further they analyzed the global stability.

Hassel (1978) [14] showed that **adding a large number refuge to a model**, in which the absence of a refuge exhibits divergent oscillations, replaces the oscillatory behavior with a stable equilibrium.

Cheng (1981) [15] published a result giving criteria for the **uniqueness of limit cycles** when they exist for a specialized class of Prey – Predator model.

Freedman and Waltman (1984) [16] considered three level food webs , **two competitive predator feeding on a single prey** and **single predator feeding on two competing prey** species . They obtained the criteria for the system to be persistent.

Kumar and Freedman (1989) [17] considered three level **food web with mutualism** and discussed the role of mutualism. They considered two cases, mutualism with prey and mutualism with first predator. They developed the persistence and extinction criteria.

Freedman and Shukla (1991)[18] developed models of single-species and predator-prey systems in a polluted closed environment. Three cases were considered: a single influx of toxicant, a constant influx of toxicant, and a periodic pollution of the environment. In the case of single-species growth they were able to determine some local and global dynamics. In the case of predator-prey systems, they investigated the existence of steady states for a small constant influx of toxicant.

Dubey and Hussain (2000) [19] analyzed the **effect of an environmental pollution** on two interacting species. The interaction between two species considered to be of three types; **competition, cooperation, and prey-predator**. In each case criteria for the local stability, instability, global stability of the non-negative equilibriums were obtained.

D. Mukherjee (2002) [20] studied **delay effect** on a model of **single species in a polluted and closed environment**. The case of constant emission and instantaneous spill of a toxicant were considered. Sufficient condition for the persistence of the population was derived.

Dubey and Upadhyay (2004)[22] proposed a model and analyzed the dynamics of one prey two predators system with **ratio-dependent predator's growth rate**. Criteria for local stability of the non-negative equilibria were obtained. The permanent co-existence of three species was also discussed.

Bohner et.al. (2006) [23] explored the **periodicity of the dynamic equations incorporated with predator prey systems and competition systems.**

Toaha and Hassan (2008) [24] studied the effect of **time delay and harvesting** on the dynamics of the predator - prey model with a time delay in the growth rate of the prey equation. The predator and prey are then harvested with constant rates. And found that the time delay can induce instability of the stable equilibrium point, Hopf bifurcation and stability switches.

Rai and Singh (2008) [25] presented a model of mutualism with four species. In this model they involved the interaction of prey, prey mutualist and two competing predators. They found the conditions for boundedness of the solution and local and global stability of the equilibriums.

Sinha et.al. (2010) [26] studied the **combined effect of environmental toxicant and disease** on prey–predator system. It was assumed in this paper that the environmental toxicant affects both prey and predator population and the infected prey was assumed to be more vulnerable to the toxicant and predation compared to the sound prey individuals.

Sinha and Dhar (2010) [27] proposed a model to study the **simultaneous effects of toxicant and infectious disease on a competing species system.** It was assumed that competing populations are adversely affected by the toxicant and one of them is vulnerable to an infectious disease. Conditions for the existence of interior equilibrium and for non-linear stability of the interior equilibrium were obtained.

Agrawal and Pathak (2011)[28] discussed the role of **additional food to common predator** on **dynamics of two competing preys**. And they developed criteria for local stability, instability and global stability of the non- negative equilibrium points using differential inequalities. They obtained sufficient conditions that ensured the persistence of the system.

Alebraheem and Abu Hasan(2011) [29] investigated the dynamical relation of **two predator species predating on a single prey** species. The approach here was slightly taken different from others the condition for the persistency which was interpreted as efficiency of search and capture, was then obtained.

Atul Johri et.al. (2012) proposed and analyzed a mathematical model to study the **diseased prey** in a LotkaVolterra Prey-Predator model.

Xu and Li (2012) [30] considered the **Beddington- De Angelis functional response** predator-prey model. The conditions for the local stability and the existence of Hopf bifurcation at the positive equilibriums were derived.

Kar and Batbayal(2012)[31] presented mathematical analysis of the dynamics of a **two prey one predator system in the presence of the time delay due to gestation.** They found criteria for the persistence of the three species and the global dynamics of the system. They showed that, the time delay may play a significant role in the stability of the system. Lastly, some numerical simulations are given to illustrate analytical results.

Chauhan and Mishra (2012) [32] proposed a model to study the **effect of pollutant and virus induced disease on single species** animal population and its essential mathematical features are analyzed. It was observed that the susceptible population does not vanish when it was only under the effect of infection but in the polluted environment, it can go to extinction.

Das et.al.(2013)[33]studied predator-prey model with **Holling type II** functional response incorporating a **constant prey refuge and independent harvesting in both species.** It was observed that constant prey refuge can cause an unstable equilibrium to become stable.

Gao and Jin(2014)[34] studied a predator-prey system that has a **discrete delay, stage structure and Beddington-De Angelis functional response**, where predator species has three stages, immature, mature and old age stage. They obtained a sufficient condition for the existence of a periodic solution.

Sinha and Chanda (2014) [35] modeled the Lotka-Volterra equations with a **time delay**. For this model they derived the conditions for the existence of steady state conditions, asymptotic stability.

Han and Li (2014) [36] investigated two species Lotka-volterra **delayed stochastic model with and without pollutant**. They showed that there exists a unique non-negative solution in each system (with and without pollutant) which was permanent in time average under certain conditions.

Naresh et.al. (2014) [37] proposed model to analyzed the **effect of toxicant on plant biomass with time delay.** They assumed that toxicant up taken by plant is converted into intermediate product after a considerable time period and this intermediate product then effect the intrinsic growth rate of the plant biomass. It was found that positive equilibrium which was locally stable without delay remain locally stable under certain conditions when the time delay is less than the threshold value otherwise become unstable.

Haung et.al. (2015) [38] studied the effects of environmental toxins on food web dynamics, they developed a toxin-dependent predator–prey model that combined both direct and indirect toxic effects on two trophic levels. They investigated how the balance of the classical predator–prey dynamics changes as a function of environmental toxin levels. While high toxin concentrations were shown to be harmful to both species, possibly leading to extirpation of both species.

Liu and Bai (2015) [39] proposed and studied a cooperative model in a polluted environment with stochastic perturbations and impulsive toxicant input. For each population, sufficient conditions for extinction, strong persistence in the mean and stochastic permanence were established.

Sinha and Kumar(2016)[40] represented a system of populations interacting in a **simple food chain and mutualist.** They considered the case of facultative mutualism between the prey at the lowest tropic level and mutualist and developed the criteria for persistence and stability.

Sayekti et.al.(2016) [41] investigated the interaction between prey, secondary predator and primary predator as a model of one prey and two predators with **constant prey harvesting**. Human invention to controlling prey population is needed and analyzed how that would effect on the existence of secondary pred-

ator and primary predator population. They found that as long as harvesting rate is smaller than intrinsic growth rate, coexistence might be achieved.

Mishra and Annavarapu (2016) [42] discussed the system level effects of a toxicant on a three species food chain system. In the models, they have assumed that the presence of top predator reduces the predatory ability of the intermediate predator. The stability analysis of the models was carried out and the sufficient conditions for the existence and extinction of the populations under the stress of toxicant were obtained.

Chauhan et.al. (2017) [43] considerd the **effect of pollution** on prey-predator model **with infected predator** and modified the model by introducing pollutant in susceptible predator and infected predator. They observed that if the rate of infection and rate of depletion of predator due to pollutant is less than some threshold then equilibrium becomes unstable and periodic solution bifurcates. Thus Hopf bifurcation occurred.

Chauhan et.al. (2018) [44] have studied mathematical model for **single species which is infected by viral disease under the influence of polluted environment**. The boundedness of the system was discussed. The existence of equilibriums and their local stability was determined. And further they discussed the occurrence of Hopf Bifurcation.

5. Proposed Study

In this research it is proposed to study the effect of pollution on the stability and long term persistence of interacting predator-prey populations using system of nonlinear differential equations under the following situations:

Model I: Predator-prey system with pollution

This is generalized Gause type predator-prey system with polluted environment. x, y are the populations of prey, predator respectively at any time t. $g(x)$is growth rate of prey population, $p(x)$ is predation rate of prey population and $s(y)$ is the death rate of predator population. c is the conversion factor converting prey biomass into predator biomass.

$$x' = xg(x) - yp(x) - r_1 C_O x$$

$$y' = y(-s(y) + cp(x) - r_2 C_O)$$

$$C_E' = -hC_E + Q_E$$

$$C_O' = a_1 C_E + \frac{d_1 \eta \phi}{a_1} - (l_1 + l_2) C_O$$

Where C_E is the concentration of toxicant in the environment, $C_O$is the concentration of the toxicant in the organism, a_1 denotes environmental toxicant uptake rate/mass of organism, d_1 the uptake rate of toxicant in food /mass of organism η is the conc. of the toxicant in the resource, ϕ the average rate of food intake/mass of organism, l_1 and l_2 are the ingestion and depuration rate of toxicant , Q_E is the toxicant input in the environment, r_1, r_2 are the rates of loss of biomass for prey, predator respectively.

Model II: Two competing prey species and single predator with pollution in the environment:

$$x_1' = x_1 g_1(x_1) - q_1 x_1 x_2 - yp_1(x_1) - r_1 C_E x_1$$

$$x_2' = x_2 g_2(x_2) - q_2 x_1 x_2 - yp_2(x_2) - r_2 C_E x_2$$

$$y' = y(-s(y) + c_1 p_1(x) + c_2 p_2(x) - r_3 C_E)$$

$x_i \; for \; i = 1, 2$ are prey species population with competition, y is predator population, s is the death rate of the predator, $g_i \; for \; i = 1, 2$ are growth rates of prey species, $p_i \; for \; i = 1, 2$ are predation functions, $r_i \; for \; i = 1, 2, 3$ are the rates of loss of biomass for three species $x_1, x_2, \; y$ respectively due to environmental pollution, $q_i \; for \; i = 1, 2$ are interspecies competition coefficients, c_i are the coefficients for conversion of prey biomass into predator biomass. $C_E$is the environmental pollution which is constant.

Model III: Single prey with two competitive predators and with pollution in the environment:

$$x' = xg(x) - yp_1(x) - yp_2(x) - r_1 C_E x$$

$$y_1' = y_1(-s_1(y_1) + c_1 p_1(x) - e_1 y_2 - r_2 C_E)$$

$$y_2' = y_2(-s_2(y_2) + c_1 p_2(x) - e_2 y_1 - r_3 C_E)$$

x is prey population, $y_i \; for \; i = 1, 2$ are predator populations with interspecies competition, $p_i \; for \; i = 1, 2$ are predation functions, $s_i for \; i = 1, 2$ are the death rates of the predators, $c_i \; for \; i = 1, 2$ are the coefficients converting prey biomass into predator biomass, $e_i \; for \; i = 1, 2$ are the interspecies competition coefficients, , $r_i \; for \; i = 1, 2, 3$ are the rates of loss of biomass for three species x, y_1, y_2 respectively due to environmental pollution, C_E is the environmental pollution which is constant.

Model IV: Predator-prey model with time delay in growth rate of prey and pollution in the environment:

$$x' = xg(x(t - \tau)) - yp(x) - r_1 C_E x$$

$$y' = y(-s(y) + cp(x) - r_2 C_E)$$

Where $g(x)$is growth rate of prey population and τ is the delay, $p(x)$ is predation rate of prey population and $s(y)$ is the death rate of predator population c is the conversion factor converting prey biomass into predator biomass.$C_E$is the environmental pollution which is constant, $r_i \; for \; i = 1, 2$ are the loss rate of biomass for prey and predator.

6. Advancement of our model from previous models:

Following are the basis on which our proposed models are advanced to previous works:

6.1. Generalized growth rate, death rate, predation rate of the species involved:

In previous similar studies the impact of pollution on the dynamics of predator prey interactions was handled by taken some specific functions for growth rate, death rate, predation etc. which are applicable only for specific species whose growth rate, death rate, predation rate are of those type. But here we have use generalize functions (Gause type model) which will be applicable for different species not for specific.

6.2. Increased dimension: In previous studies similar models formed for two species but we developed models for three species.

6.3. Increased interaction type: In previous studies the models formed with single type of interaction either predation or competition. But in our models we have taken two types of interactions predation and competition at the same time. It makes our models more realistic.

S. No.	Author (year)	Type of interaction	Dimension	Growth, death, predation rate functions
1.	Huaping L., Z. Ma (1990)	Competition, predation	2	Specific functions
2.	Z. Li, S. Zhisheng, K. Wang (2004)	Competition	1	Generalized functions
3.	H. I. Freedman J. B. Shukla (1991)	competition	1	Generalized functions
4.	B. Dubey J. Hussain (2000)	Predation	2	Specific functions
5.	S. Sinha, J. Dhar O. P. Mishra (2010)	Predation	2	Specific functions
6.	S. Sinha J. Dhar (2010)	Competition	2	Specific functions
7.	S. Chauhan O. P. Mishra (2012)	Competition	1	Specific functions
8.	S. Chauhan, S. K. Bhatia P.Chaudhary (2017)	Predation	2	Specific functions
9.	S. Chauhan, S. K. Bhatia and S. Shrma (2018)	Competition	1	Specific functions
10.	Our proposed models			
	i.	Predation	2	Generalized Functions
	ii.	Predation, competition	3	Generalized Functions
	iii.	Predation, competition	3	Generalized Functions
	iv.	Predation	2	Generalized Functions

7. Research Methodology

7.1. Study of related literature.

7.2. Developing a nonlinear differential equation prey predator model with environmental pollution.

7.3. Analysis of the model.

7.3.1. Testing the existence of its solution.

7.3.2. Ensuring the existence and uniqueness of its solution.

7.3.3. Evaluating the possibility of existence of its equilibrium points.

7.3.4. Setting criteria for their stability (Local and Global both).

7.4. Setting criterion for the long term existence of the model.

7.5. Example through Computer Simulations.

7.6. Conclusion.

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